Midterm Semester-2015

Solution 1. Let $y(t)$ be the displacement and $v(t)$ be the velocity of the object at time t. Here initial conditions $y(0) = 0$ and $v(0) = 0$. Now the only force guiding the object is gravity. Therefore

$$\frac{dv(t)}{dt} = g \tag{1}$$

and also

$$\frac{dy(t)}{dt} = v(t). \tag{2}$$

Let $f : [0, \infty] \to \mathbb{R}$ be the height of the object above the ground level. Then

$$f(t) = 300 - y(t).$$

Therefore from (1) and (2), we have

$$\frac{df(t)}{dt} = -\frac{dy(t)}{dt} = -v(t), \tag{3}$$

$$\frac{d^2 f(t)}{dt^2} = -\frac{dv(t)}{dt} = -g$$

and $f(0) = 300$.
This is the required IVP.

Using (1) and integrating (3), we have

$$\int_{300}^{f(t)} df(t) = \int_0^t -gtdt.$$

Therefore

$$f(t) = 300 - \frac{1}{2}gt^2.$$

Now $f(T) = 0$ implies $gT^2 = 600$. This is the required equation.

Solution 2 (a) **.** Let $F(x, y(x)) = y|y|$, $y(x_0) = y_0$ and $G(x, y(x)) = y^{\frac{1}{3}} + x$, $y(x_0) = y_0$. Clearly, F and G are continuous in $\mathbb{R}^2$. So by Cauchy Peano theorem for every $(x_0, y_0) \in \mathbb{R}^2$, two IVPs

$$y' = F(x, y(x)), y(x_0) = y_0$$

and

$$y' = G(x, y(x)), y(x_0) = y_0$$

have a local solution at x_0.

(b) Here $F(x, y(x)) = y|y|$ is continuous in $\mathbb{R}^2$ and differential in x. Also $\frac{\partial F}{\partial y} = 2|y|$, which is bounded around any bounded neighbourhood of any $(x_0, y_0) \in \mathbb{R}^2$. This implies that F is Lipschitz in y. So by Picard Lindelof theorem, for every $(x_0, y_0) \in \mathbb{R}^2$, there exists a unique local solution at x_0.

Similarly, $G(x, y(x)) = y^{\frac{1}{3}} + x$ is continuous in $\mathbb{R}^2$ and differential in x. Also

$$\frac{\partial G}{\partial y} = \frac{1}{3y^{\frac{2}{3}}}.$$

If $y_0 \neq 0$ then any bounded neighbourhood of any y_0 not containing 0, $\frac{\partial G}{\partial y}$ is bounded. So by Picard Lindelof theorem, for every $(x_0, y_0) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$, there exists a unique local solution at x_0. If $y_0 = 0$, then G is not Lipschitz in y in any neighbourhood around $(x_0, 0)$ for any $x_0 \in \mathbb{R}$. So there is no unique solution in this case.

Solution 3**.** Suppose $\mu = e^{\int g(z)dz}$ is an integrating factor for

$$M(x, y)dx + N(x, y)dy = 0.$$

Then

$$\frac{\partial \tilde{M}(x, y)}{\partial y} = \frac{\partial \tilde{N}(x, y)}{\partial x},$$

where $\tilde{M}(x, y) = e^{\int g(z)dz} M(x, y)$, $\tilde{N}(x, y) = e^{\int g(z)dz} N(x, y)$ and $z = xy$. Now

$$\frac{\partial \tilde{M}(x, y)}{\partial y} = \mu \frac{\partial M(x, y)}{\partial y} + \mu x g(z) M(x, y) \tag{4}$$

and

$$\frac{\partial \tilde{N}(x, y)}{\partial x} = \mu \frac{\partial N(x, y)}{\partial x} + \mu y g(z) N(x, y). \tag{5}$$

Equating (4) and (5), we have

$$\frac{\partial M(x, y)}{\partial y} + x g(z) M(x, y) = \frac{\partial N(x, y)}{\partial x} + y g(z) N(x, y).$$

Therefore

$$g(z) = \frac{\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}}{Ny - Mx}.$$

This completes the proof.

Solution 4 (a)**.** It is enough to prove that Wronskian either vanishes for all values of x or it is never vanishes. Let $y_i, 1 \leq i \leq n$ be the solution of the nth order homogeneous differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y^{(1)} + p_n(x)y = 0 \tag{6}$$

We will conssider only for $n = 3$. We can rewrite eq. (6) as a first order matrix differential equation. Defining the vector

$$\mathbf{Y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

Then

$$\frac{d\mathbf{Y}}{dx} = B(x)\mathbf{Y}, \tag{7}$$

where the matrix $B(x)$ is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_3(x) & -p_2(x) & -p_1(x) \end{pmatrix}$$

Now if y_i, for $1 \leq i \leq 3$ are linearly independent solutions to (6) (for $n = 3$), then the matrix $A_{n,n}$ satisfies the first order matrix differential equation,

$$\frac{dA_{n,n}}{dx} = B(x)\mathbf{Y}$$

Now taking derivative of determinant of the matrix $A_{n,n}$, we have in term of Wronskian $\mathbf{W}$,

$$\frac{d\mathbf{W}}{dx} = \mathbf{W}\ TrB(x),$$

where $TrB(x) = -p_1(x)$, trace of the matrix $B(x)$. Integrating from a to x, we have

$$\mathbf{W(x)} = \mathbf{W(a)}\exp\{-\int_a^x p_1(t)dt\}$$

Note that if $\mathbf{W(a)} \neq 0$, then $\mathbf{W(x)} \neq 0$ for any x. If $\mathbf{W(a)} = 0$, then $\mathbf{W(x)} = 0$ for all x. This confirms our assertion that the Wronskian either vanishes for all values or it is never equal to zero. Similarly, one can show this for $n \geq 4$.

Solution 4 (b)**.** Here we will prove only for $n = 2$. That is if y_1 and y_2 are two linearly independent solutions of the differential equation

$$y^{(2)} + p_1(x)y^{(1)} + p_2(x)y = 0 \tag{8}$$

then any function y_g satisfying (8) is a linear combination of y_1 and y_2. Suppose y_g satisfying (8) is not a linear combination of y_1 and y_2. Then from Wronskian, we have

$$\mathbf{W}(y_1, y_2; x) = c_1\exp\{-\int^x p_1(t)dt\} \tag{9}$$

$$\mathbf{W}(y_1, y_g; x) = c_2\exp\{-\int^x p_1(t)dt\} \tag{10}$$

$$\mathbf{W}(y_2, y_g; x) = c_3\exp\{-\int^x p_1(t)dt\} \tag{11}$$

where c_1, c_2 and c_3 are arbitrary constants. Now multiplying the above equations y_3, y_2 and y_1 respectively and adding we have

$$y_3\mathbf{W}(y_1, y_2; x) + y_2\mathbf{W}(y_1, y_g; x) + y_1\mathbf{W}(y_2, y_g; x) = (c_1y_3 + c_2y_2 + c_3y_1)\exp\{-\int^x p_1(t)dt\}. \tag{12}$$

We have from the above

$$c_1y_3 + c_2y_2 + c_3y_1 = 0.$$

Since y_1 and y_2 are linearly independent solutions of (8), and hence $\mathbf{W}(y_1, y_2; x) \neq 0$. Therefore $c_1 \neq 0$. This implies

$$y_3 = -\frac{c_2}{c_1} y_2 - \frac{c_3}{c_1} y_1.$$

Hence our assumption is false. This completes the proof.

Solution 5. Let $y = \sum\limits_{n=0}^{\infty} a_n x^n$. Now substituting $y^{''}, y^{'}$ and y to the differential equation

$$y^{''} - 2xy^{'} + \lambda y = 0,$$

we have

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (2n-\lambda)a_n] x^n = 0.$$

Therefore

$$a_{n+2} = \frac{(2n-\lambda)a_n}{(n+1)(n+2)}, \text{ for all } \ n \geq 0.$$

If n is even, then

$$\begin{aligned} a_n &= \frac{(2n-4-\lambda)a_{n-2}}{(n)(n-1)} = \frac{(2n-4-\lambda)(2n-8-\lambda)a_{n-4}}{(n)(n-1)(n-2)(n-3)} \\ &= \ldots = \frac{\prod\limits_{k=1}^{\frac{n}{2}} (2n-4k-\lambda)a_0}{n!}. \end{aligned} \tag{13}$$

Similarly for n is odd,

$$\begin{aligned} a_n &= \frac{(2n-4-\lambda)a_{n-2}}{(n)(n-1)} = \frac{(2n-4-\lambda)(2n-8-\lambda)a_{n-4}}{(n)(n-1)(n-2)(n-3)} \\ &= \ldots = \frac{\prod\limits_{k=1}^{\frac{n-1}{2}} (2n-4k-\lambda)a_1}{n!}. \end{aligned} \tag{14}$$

Therefore $y = \sum\limits_{n=0}^{\infty} a_n x^n$ is a solution with the coefficient a_n is given in (13) and (14).

If $a_1 = 0$, then we get $y_1 = \sum\limits_{n=0}^{\infty} a_n x^n$, where for even n, a_n is given in (13) and $a_n = 0$ for odd n.

If $a_0 = 0$, then we get $y_2 = \sum\limits_{n=0}^{\infty} a_n x^n$, where for odd n, a_n is given in (14) and $a_n = 0$ for even n. These two solutions, y_1 and y_2, are linerly independent.

If one of the solution has to be polynomial, a_n has to be zero except for finite numbers. If $\lambda \in 2\mathbb{N}$, one of the solutions would be a polynomial.

Solution $6(a)$**.** In normal form, Bessel's equation is

$$u^{''} + \left(1 + \tfrac{1-4p^2}{4x^2}\right) u = 0.$$

We will use the following theorem:

Suppose that q and $\tilde{q}$ are positive functions with $q > \tilde{q}$. Let y be a nontrivial solution of the differential equation

$$y^{''} + qy = 0$$

and let $\tilde{y}$ be a nontrivial solution of the differential equation

$$\tilde{y}^{''} + \tilde{q}\tilde{y} = O.$$

Then y vanishes at least once between any two successive zeros of $\tilde{y}$.

If $0 < p < \frac{1}{2}$ then

$$1 + \frac{1 - 4p^2}{4x^2} > 1$$

and if $p > \frac{1}{2}$ then

$$1 + \frac{1 - 4p^2}{4x^2} < 1.$$

Now using the above theorem to Bessel's equation and to $y^{''} + y = 0$ The assertions follow.

Solution $6(b)$. We say that a singular point x_0 for the differential equation

$$y^{''} + py' + qy = 0$$

is a regular singular point if $(x - x_o)p(x)$ and $(x - x_o)q(x)$ are analytic at x_0. Solution of the form

$$y = y(x) = x^r(a_o + a_1x + a_zx_2 + \cdots),$$

where $a_0 \neq 0$ and r is any real number, of a differential equation at a regular singular point x_0 is said to a Frobenius series solution

For any $p \geq 0$, one Frobenius series solution (namely, for larger root of the indicial equation) is guaranteed.

Solution $6(c)$. For $p = 1$, the Bessel's equation is

$$x^2y^{''} + xy' + (x^2 - 1)y = 0. \tag{15}$$

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, r is any real number. Substituting $y, y^{'}$ and $y^{''}$ to (15), we have indicial equation

$$r^2 - 1 = 0.$$

The roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Also they differ by a positive integer. The recurrence relation is

$$a_n = -\frac{a_{n-2}}{(n+1)^2 - 1} \tag{16}$$

for $n = 2, 3, \ldots$ Also note that for $r_1 = 1$, $a_1 = 0$ and hence all the odd coefficients, i.e. $a_{2n+1} = 0$ for $n = 1, 2, \ldots$. Therefore from (16),

$$a_{2m} = -\frac{a_{2m-2}}{(m+1)^2 - 1} = (-1)^m \frac{a_0}{2^{2m}(m+1)!m!}$$

for $m = 1, 2, 3, \ldots$.

Therefore the Frobenius series solution of the differential equation is

$$y_1 = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}(m+1)!m!} x^{2m} \right]$$

for $x > 0$.